

TPI-MINN-99/41-T
UMN-TH-1816/99
ITEP-TH-44/99
hep-th/9910071

BPS Saturated Solitons in $\mathcal{N} = 2$ Two-Dimensional Theories on $R \times S$ (Domain Walls in Theories with Compactified Dimensions)

Xinrui Hou¹⁾, A. Losev^{1,2)}, and M. Shifman¹⁾

¹⁾ *Theoretical Physics Institute, Univ. of Minnesota, Minneapolis, MN 55455*

²⁾ *Institute of Theoretical and Experimental Physics, Moscow 117259, Russia[†]*

Abstract

We discuss topologically stable solitons in two-dimensional theories with the extended supersymmetry assuming that the spatial coordinate is compact. This problem arises in the consideration of the domain walls in the popular theories with compactified extra dimensions. Contrary to naive expectations, it is shown that the solitons on the cylinder can be BPS saturated. In the case of one chiral superfield, a complete theory of the BPS saturated solitons is worked out. We describe the classical solutions of the BPS equations. Depending on the choice of the Kähler metric, the number of such solutions can be arbitrarily large. Although the property of the BPS saturation is preserved order by order in perturbation theory, nonperturbative effects eliminate the majority of the classical BPS states upon passing to the quantum level. The number of the quantum BPS states is found. It is shown that the $\mathcal{N} = 2$ field theory includes an auxiliary $\mathcal{N} = 1$ quantum mechanics, Witten's index of which counts the number of the BPS particles.

[†] Permanent address

1 Introduction and Physical Motivation

The idea that our matter is made of zero modes trapped on the surface of a (1+3)-dimensional topological defect (domain wall) embedded in a higher-dimension universe dates back to 1983 [1]. With the advent of supersymmetry and Planckian physics, it was natural to attempt to exploit [2] the setup for solving a wide range of questions, such as building a fundamental theory at the scale much below Planckian, developing an appropriate pattern of supersymmetry breaking, and so on. The BPS saturated domain walls which preserve a part of the original supersymmetry, play a special role. It was shown [2, 3] that the scalar and spinor matter, as well as the gauge fields, can be localized on (1+3)-dimensional dynamical walls. The next step was made in [4] where gravity was included in consideration. Since the gravitons are not localizable on dynamical walls, it was suggested that the original multidimensional space-time is compact with respect to one or more coordinates, it has the structure of a cylinder $S^k \times \mathcal{M}_4$ ($k \geq 1$), and the gravitons propagate in the bulk. All other fields are still localized on the wall that appears dynamically on the cylinder. The approach was later dubbed *large extra dimension(s) theories*. Phenomenologically it is preferable [4] to have $k = 2$, the wall width of order, roughly, 1 (TeV)^{-1} , and the radius of the cylinder of order 1 mm. For a recent discussion of the emerging quite rich phenomenology see e.g. [5].

Surprisingly, dynamical aspects of this construction have been investigated at a rather fragmentary level. In particular, the question of interest is the issue of the BPS saturation of the wall-like topological defects on the cylinder, which has never been addressed in full previously. Studying such topological objects is the task of this work. Since the domain wall is a static field configuration depending on a single coordinate, while other spatial coordinates are passive, this problem in many aspects is equivalent to studying BPS saturated solitons in two-dimensional theories obtained by dimensional reduction of multidimensional theories. In other words, one starts from D -dimensional theory, in which $D - 1$ coordinates are spatial. The wall solution depends on one of them. The presence of “extra” $D - 2$ spatial dimensions is irrelevant at the classical level (although it may be relevant at the quantum level). The extra $D - 2$ dimensions can be reduced. The two-dimensional theory obtained in this way has an extended supersymmetry (SUSY). We will consider here the generalized Wess-Zumino models, also referred to as the Landau-Ginzburg theories, in two dimensions.

For instance, if one starts from (1+3)-dimensional Wess-Zumino model with a discrete set of SUSY vacua, the walls are two-dimensional objects (two space dimensions plus time) which can be assumed to lie in the yz plane. The wall profile depends on x . If the original theory is considered in a noncompact space, \mathcal{M}_4 , the topological stability of the wall is achieved in a rather trivial manner. Let Φ_{*1} and Φ_{*2} be two distinct degenerate vacua of the theory, $V(\Phi_{*1}) = V(\Phi_{*2}) = 0$, where V is the scalar potential. Then, the minimal energy solution $\Phi(x)$ interpolating between Φ_{*1} at $x = -\infty$ and Φ_{*2} at $x = +\infty$ is topologically stable. (Here x is

the spatial coordinate.) Such domain walls (solitons) always exist. They may or may not be BPS saturated. The issue of the BPS saturation requires a separate dynamical consideration [6]. A general theory of the BPS saturated solitons in the $\mathcal{N} = 2$ Landau-Ginzburg models in this case was worked out in [7].

If the world sheet is a cylinder, $R \times S$, the situation with the topological stability is different. Indeed, x now becomes a compact coordinate. If the radius of S is r , the points x and $x + 2\pi r$ are identified (we will also use the notation $L = 2\pi r$ and will often put $r = (2\pi)^{-1}$ in what follows, so that $L = 1$). To produce topologically stable solitons, the dynamical theory under consideration must be such that the field Φ is defined on a manifold M with a noncontractible cycle or several distinct cycles. If, as one winds around S , i.e. x varies continuously from $x = 0$ to $x = L$, the field Φ winds around a cycle of M , the corresponding field configuration $\Phi(x)$ will be topologically stable [2]. Unlike the theories on noncompact manifolds, the topologically stable configuration described above need not be related to the vacua of the theory. The latter may not exist at all (the run-away theories). Thus, the issue reduces to the classification of noncontractible closed contours ¹ on M , i.e. the fundamental group $\pi_1(M)$.

Let us assume that a topologically stable soliton exists. Can it be BPS saturated? As is well-known, the issue of the saturation is related to the existence of the central charge(s) Z in the superalgebra [8]. The nonvanishing central charge is a necessary condition for the soliton to be saturated. Since the spatial coordinate is compact in our problem, $\Phi(0) = \Phi(L)$, and the superpotential W is a holomorphic function of Φ , at first sight one might conclude that $Z = |W[\Phi(0)] - W[\Phi(L)]| = 0$, and the saturated solitons are not possible. In fact, this conclusion is wrong. It is the differential dW that must be a single-valued function (it determines the scalar potential), the superpotential need not be single-valued. The superpotential W can be a locally holomorphic function with branches. Then the central charges are determined by the integrals of dW over various noncontractible cycles on M , i.e. by periods of the differential dW ,

$$Z_i = \int_{\text{nc cycle}_i} dW. \quad (1)$$

The nonvanishing central charge is a necessary but not sufficient condition for the existence of the BPS saturated soliton. We need to make two more steps. First, we need to look for the static classical configurations that satisfy the BPS equations. Then, we need to take into account quantum effects. If the classical BPS configuration is isolated, one can construct a quasiclassical state around such configuration. The perturbative quantum corrections would change the shape of this quasiclassical state, but at the level of perturbative quantization, the modified states would still be annihilated by half of the supercharges. At the same time, due to nonperturbative quantum effects, not all these states may survive in the

¹The space of all loops in M could be divided into classes such that loops inside each class can be smoothly deformed into each other. Such space of classes is denoted by $\pi_1(M)$.

nonperturbative quantum theory – due to instanton-like phenomena, the half of the supercharges that acted trivially at the classical level, may in fact start acting nontrivially, connecting these states. Then, such states cease to be BPS saturated. This phenomenon was first observed by Witten [9] in the interpretation of the Morse theory in terms of $\mathcal{N} = 1$ supersymmetric quantum mechanics. The critical points of the superpotential were quasiclassical “BPS configurations”, and instantons provide mixings between them.

The paper is organized as follows. In Sec. 2 we will study the space of the classical BPS configurations and present a heuristic evaluation of the number of the BPS states in the generalized Landau-Ginzburg models based on the properties of Cecotti–Fendley–Intriligator–Vafa (CFIV) index [10]. We will show that only vicinities of the poles of dW contribute to the BPS solitons at the quantum level.

In the Sec. 3 we will illustrate the general consideration of Sec. 2 by a typical example. In the Sec. 4 we clarify the assertions made in Sec. 2, without the use of the CFIV index. We show that the $\mathcal{N} = 2$ supersymmetric field theory contains within it an auxiliary $\mathcal{N} = 1$ quantum mechanics, so that Witten’s index of the latter counts the number of the BPS solitonic states in the former. *En route* we show that the “localization to the poles of dW ” phenomenon has analogs in the $\mathcal{N} = 2$ supersymmetric quantum mechanics on the space with freely acting isometries (the nonvanishing Killing vectors).

2 A Heuristic Derivation of the Number of BPS States in the Generalized Landau–Ginzburg Model

2.1 Generalized Landau-Ginzburg Model and the Classical BPS Equations

The action of the generalized Landau-Ginzburg (GLG) model, to be considered below, has the form

$$S = \frac{1}{4} \int d^2x d^4\theta K(\Phi^i, \bar{\Phi}^{\bar{j}}) + \left\{ \frac{1}{2} \int d^2x d^2\theta W(\Phi^i) + \text{H.c.} \right\}, \quad (2)$$

where $x^\mu = \{it, x\}$ ($\mu = 0, 1$) and $x \in S$. Moreover, Φ^i is a set of the chiral superfields (corresponding to the holomorphic coordinates on the target space M), the superpotential W is a (multivalued) analytic function of all chiral variables Φ^i , while the kinetic term is determined by the Kähler potential K , which is a real (multivalued) function depending both on chiral Φ^i and antichiral $\bar{\Phi}^{\bar{j}}$ fields.

While W and K are multivalued, the Kähler metric,

$$G_{i\bar{j}} = \frac{\partial^2 K}{\partial \Phi^i \partial \bar{\Phi}^{\bar{j}}} \quad (3)$$

and the 1-differential $\Omega = \Omega_i d\Phi^i$,

$$\Omega_i(\Phi) = \frac{\partial W}{\partial \Phi^i} \quad (4)$$

are single-valued on the target space M . This is a necessary requirement, to be imposed here and below. It ensures that the scalar potential and the fermion terms in the Lagrangian are well-defined.

In components the Lagrangian takes the form

$$\mathcal{L} = \sum_{i,\bar{j}=1}^n \left\{ G_{i\bar{j}} \partial_\mu \Phi^i \partial_\mu \bar{\Phi}^{\bar{j}} + G^{i\bar{j}} \frac{\partial W}{\partial \Phi^i} \frac{\partial \bar{W}}{\partial \bar{\Phi}^{\bar{j}}} \right\} + \text{fermions}, \quad (5)$$

where n is the number of the chiral (antichiral) fields involved, and $G^{i\bar{j}}$ is the inverse matrix,

$$G_{i\bar{j}} G^{k\bar{j}} = \delta_i^k.$$

The equations of the BPS saturation have the form

$$\dot{\Phi}^i = e^{i\delta} G^{i\bar{j}} \frac{\partial \bar{W}}{\partial \bar{\Phi}^{\bar{j}}}, \quad \dot{\bar{\Phi}}^{\bar{j}} = e^{-i\delta} G^{i\bar{j}} \frac{\partial W}{\partial \Phi^i}, \quad (6)$$

where the dot denotes differentiation over the *spatial* coordinate x . Let us denote by $\Gamma \in M$ the loop in M that is the image of the map Φ . Then, the phase δ appearing in Eq. (6) is that of the period,²

$$\int_\Gamma \Omega \equiv \Delta_i W \equiv e^{i\delta} |\Delta_i W|. \quad (7)$$

The formula (6) must be viewed as the master equation.

2.2 How to Solve the BPS Equations for the One-Dimensional Target Space

The general solution of Eq. (6) for the one-dimensional target space could be divided into two steps. At the first step we will find the space \mathcal{S} of solutions $\Phi(x, s)$ of Eq. (6) (s is the coordinate on the space \mathcal{S}), which satisfy a modified periodicity condition,

$$\Phi(0, s) = \Phi(l(s), s), \quad (8)$$

for some function $l(s)$.

Then, we will pick up the proper elements from \mathcal{S} , the classical BPS solitons, by imposing the condition

$$l(s) = L. \quad (9)$$

²The period integral depends only on the class of Γ in $\pi_1(M)$.

The requirement (9) determines (generally speaking) a discrete set of the parameters $\{s_i\}$ for which one deals with the classical BPS solutions.

Let us start from the first step. The space \mathcal{S} could be considered as a space of such closed unparametrized curves on M that:

(i) These curves are tangent to the vector field with the components $(\text{Re}V, \text{Im}V)$,

$$V = \frac{1}{G} e^{i\delta} \frac{\partial \bar{W}}{\partial \bar{\Phi}},$$

where $G = \partial^2 K / \partial \Phi \partial \bar{\Phi}$;

(ii) These curves do not touch the points where the vector field V vanishes (critical points of V).

Then, let us take such a closed curve Γ_s , and pick up some parametrization y on it, so that the points on Γ_s have coordinates $\Phi_s(y)$, $0 < y < 1$. Now, we want to find such a change of parametrization $y(x)$ that

$$\Phi_s(y(x)) \equiv \Phi(x, s) \tag{10}$$

solves Eq. (6). Substituting (10) into (6) we get

$$\frac{dx(y)}{dy} = G(\Phi_s(y), \bar{\Phi}_s(y)) \frac{d\Phi_s}{dy} e^{-i\delta} \left(\frac{d\bar{W}}{d\bar{\Phi}} \Big|_{\Phi=\Phi_s(y)} \right)^{-1}. \tag{11}$$

The right-hand side of (11) is defined and positive if and only if the curve corresponding to Φ_s is tangent to the vector field V and does not touch its critical points (where the right-hand side of (11) goes to $+\infty$). In this case we can integrate (11) to get a monotonous function $x(y)$. The function $l(s)$ is given by the integral of the right-hand side from 0 to 1,

$$l(s) = \int_0^1 dy G(\Phi_s(y), \bar{\Phi}_s(y)) \frac{d\Phi_s}{dy} e^{-i\delta} \left(\frac{d\bar{W}}{d\bar{\Phi}} \Big|_{\Phi=\Phi_s(y)} \right)^{-1}. \tag{12}$$

Here, we set $y(0) = 0$, $y(l(s)) = 1$. Note, that the space \mathcal{S} does not depend on the nondegenerate changes of metric – such changes only affect the function $l(s)$.

Now, let us discuss the structure of \mathcal{S} in more detail. The following fact is very helpful – the vector field V is orthogonal to the gradient of the multivalued function $I(\Phi, \bar{\Phi})$,

$$I = \frac{e^{-i\delta} W[\Phi(x)] - e^{i\delta} \bar{W}[\bar{\Phi}(x)]}{2i}, \tag{13}$$

which is actually the integral of motion of Eq. (6),

$$\dot{I} = \frac{1}{2i} \left(\frac{\partial W}{\partial \Phi} e^{-i\delta} \dot{\Phi} - \frac{\partial \bar{W}}{\partial \bar{\Phi}} e^{i\delta} \dot{\bar{\Phi}} \right) = 0.$$

(Note, that expression (13) is the integral of motion in the multidimensional problem, with the arbitrary number of fields $\Phi^i, \bar{\Phi}^{\bar{j}}$.)

Therefore, the closed curves tangent to V present a net of the level lines of I , i.e. the curves Γ_s are described by the equation

$$I(\Phi_s) = s. \quad (14)$$

At the same time, for the given s , the space of solutions of Eq. (14) could have several components, and not all of them would correspond to closed curves – some of the components could even be noncompact (see Sec. 3 for more details).

Now, we can describe \mathcal{S} as the space of pairs (s, a) where s is such that Eq. (14) contains at least one component that is a closed curve, and the index a just numerates such components. From the description above we conclude that the space \mathcal{S} is an open space.

In fact, for a pair (s, a) , and the corresponding curve $\Phi_{s,a}(y)$, we construct its deformation $(s + \Delta s)$, $\Delta\Phi_{s,a}$,

$$\frac{\partial I}{\partial \Phi} \Delta\Phi_{s,a} + \frac{\partial I}{\partial \bar{\Phi}} \Delta\bar{\Phi}_{s,a} = \Delta s. \quad (15)$$

Equation (15) can always be solved (for small Δs) because the closed curves do not touch the space of zeroes of the gradient of I (that coincides with the space of zeros of dW). Therefore, the allowed values of s is a union of finite intervals (s_k, s_{k+1}) , and, possibly, semi-infinite intervals $(-\infty, s_i)$ or $(s_f, +\infty)$. It is also possible that the allowed values of s form the full line $(-\infty, \infty)$.

To complete the description of the space \mathcal{S} we should understand what happens with the closed curve when s reaches its critical values. If M is compact, all critical values s_k have to be finite, and the critical curve $\Phi_{s_k,a}(y)$ is a curve passing through one of the critical points of dW .

In fact, as one can see from Eq. (12), when the curve approaches a critical point of dW , the derivative $d\bar{W}/d\bar{\Phi}$ in the denominator vanishes, so the integrand (and, thus, the integral $l(s)$) tends to $+\infty$.

If M is noncompact, we can think of it as of the space \bar{M} with several points P_α deleted. Now, we will analytically continue dW to the points P_α keeping in mind examining the behavior in these points.

If dW could be analytically continued to the point P_α , we could just add all such points to the space M to get the space M_1 . Suppose that there is a closed curve close to the point P_α – then we expect that there is also a critical closed curve, with the critical value s_c passing through the point P_α . We expect that $l(s)$ goes to infinity as s tends to s_c . From the standpoint of M , the critical curve starts at infinity and ends there. (If $l(s_c)$ is finite this means that the manifold M is geodesically noncomplete; we discard this possibility as an obvious pathology.)

Another case to be considered corresponds to dW having a pole at P_α . Then, when s tends to one of the infinities, the whole curve $\Phi_{s,a}$ runs away towards the

point P_α . If the metric G can be smoothly continued to P_α then $l(s)$ tends to zero as s tends to the corresponding infinity. (Note, that if the metric G is also singular at the point P_α , we have a competition, but we will not discuss such cases of competition here.)

After the structure of \mathcal{S} is determined, we complete the first step of our search for the classical BPS configurations. The second step is to solve Eq. (9) on \mathcal{S} . Note, that the function $l(s)$ depends on the choice of metric G , which can be taken to be an arbitrary smooth nonvanishing function. Therefore, the behavior of $l(s)$ inside the interval (s_k, s_{k+1}) is absolutely nonuniversal and can be chosen at will with the appropriate choice of G . Nevertheless, we understand the asymptotics of $l(s)$ when s tends to its critical values.

From the analysis above we see that if the critical value is finite, $l(s)$ tends to $+\infty$. If the critical value is infinite and the scalar potential tends to infinity, then $l(s)$ tends to zero.

This knowledge is sufficient to provide some information about the space of solutions of Eq. (9) which determines the classical BPS configurations. This equation has *even* number of solutions on finite intervals, *odd* number of solutions on semi-infinite intervals $(-\infty, s_k)$ and $(s_m, +\infty)$, and *even* number of solutions in the special case when the interval is $(-\infty, +\infty)$. Note, that if the scalar potential does not grow at infinities, we do not know the number of solutions even modulo 2.

2.3 Getting the Quantum BPS States from the Classical BPS Solutions

The quantum BPS solitons of the two-dimensional theory correspond to such one-particle states that are annihilated by half of the supercharges. It will be assumed that the soliton particle is at rest, i.e. the states in the quantum theory with the zero spatial momentum will be considered.

The BPS particle states are annihilated by half of the supercharges, and thus, form doublet representations (short multiplets) of the supersymmetry algebra [11]. The fact the supermultiplet of the BPS solitons contains two states can be readily seen within the quasiclassical quantization. A regular representation of $\mathcal{N} = 2$ superalgebra in 1+1 dimensions is quadruplet – it contains two bosonic and two fermionic states. This is seen, for instance, from the inspection of the chiral superfield $\Phi(x_L, \theta)$. This can also be directly inferred from the analysis of non-BPS solitons in the quasiclassical approximation. Indeed, such soliton is characterized by the following collective coordinates: the soliton center x_0 and four (complex) fermion collective coordinates $\eta_{1,2}$ and $\bar{\eta}_{1,2}$, reflecting the nontrivial action of all four supercharges when applied to the bosonic solution. Upon quantization, the collective coordinates are to be treated as (adiabatically) varying functions of time $x_0(t)$, $\eta_{1,2}(t)$, $\bar{\eta}_{1,2}(t)$. The quantum-mechanical (first quantized) Lagrangian takes the form

$$\mathcal{L} = m\dot{x}_0^2 + im\bar{\eta}_j\dot{\eta}_j, \quad j = 1, 2. \quad (16)$$

where m is the soliton mass, and the dot stands for the time derivative. If $\eta_{1,2}$ are the canonic coordinates, then $i\bar{\eta}_{1,2}$ are the conjugate canonic momenta, which determines the commutation relations

$$\{\eta_i(t), \bar{\eta}_j(t)\} = \delta_{ij}. \quad (17)$$

The latter have a matrix representation in terms of four-by-four matrices, the Hamiltonian is a four-by-four matrix, i.e. the dimension of the supermultiplet is four. What changes upon the transition to the BPS saturated soliton? Two out of four supercharges annihilate the soliton. Correspondingly, only two supercharges act non-trivially, and there are two fermionic collective coordinates, η and $\bar{\eta}$. Again η is to be treated as the canonic coordinate, $i\bar{\eta}$ is the conjugate momentum. The commutation relation which ensues is realized in terms of two-by-two matrices ($\eta \sim \sigma_-$, $\bar{\eta} \sim \sigma_+$). The dimension of the supermultiplet is two. This is the so-called shortened multiplet consisting of one bosonic and one fermionic soliton. The very existence of the shortened multiplets is due to the central extension of the original superalgebra, $Z \neq 0$.

It is well-known that quantization around the classical BPS configurations results in the BPS state in the *perturbative* quantum theory. In other words, every classical solution $\Phi(x, s_*)$, where s_* is determined from Eq. (9), gives rise to a BPS quantum state order by order in perturbation theory. This is so because of the multiplet shortening. Naively, assuming that nonperturbative corrections to the perturbative quantization do not change this, we get quite a weird picture of the BPS states in the model under consideration. Their number is arbitrary, depending on details of the metric G – even small variations of the metric could lead to the appearance of new states or disappearance of the previously existing ones (see Sec. 3 for details).

Moreover, there is an analog of Witten's index [12] that counts the number of doublets in the centrally extended $\mathcal{N} = 2$ superalgebra – the Cecotti–Fendley–Intriligator–Vafa index. This index is known to be independent of the metric (see Appendix), which contradicts our naive conclusion, based on the perturbative quasiclassical quantization, that the number of the quantum BPS states equals the number of the classical BPS solutions.

What saves the day is the observation that the number of the classical BPS configuration modulo 2 is independent of the metric. This follows from the analysis in the previous subsection. Thus, we have to conclude that nonperturbative quantum effects (like instantons in Witten's quantum mechanics) lift the BPS saturation of the classical solutions and change the structure of the representations. Shortened multiplets pair up; a pair of the doublet representations can form a regular quadruplet (non-BPS) representation. The supercharges which acted trivially at the perturbative level start acting between the states that correspond to the classical BPS configurations, making quadruplets from pairs of doublets. That is why the number of doublets in the full *quantum theory* equals the number of quasiclassical doublets only mod 2. Pairing up of two classical BPS solutions giving rise to a non-BPS quadruplet of quantum states can occur at strong coupling.

The mass shift from the BPS bound is determined by the action of the field configuration $\Phi(t, x)$ which smoothly interpolates between the two given classical BPS solutions $\Phi_1(x)$ and $\Phi_2(x)$, which pair up together. With the appropriate choice of the metric, the barrier in the space of fields separating the two classical solutions, can be made high. Correspondingly, although the BPS saturation will be lifted at the (nonperturbative) quantum level, the mass of the quadruplet representation will be different from the BPS bound (i.e. from the central charge) only exponentially. We should also note that in the problem of the domain walls (as opposed to solitons in 1+1 dimensions), the action is proportional, in addition, to the wall area. Therefore, the classical BPS wall with the infinite area remains BPS at the quantum level too, there is no tunneling.

Finally, we note that using the independence of the CFIV index on the metric, we can show that the finite intervals do not contribute to the CFIV index, while each semi-infinite interval gives contribution equal to 1. In fact, rescaling $G \rightarrow \lambda G$, we change the function $l(s)$ to $\lambda l(s)$. Let l_0 be the minimum of $l(s)$ on the finite interval. Being an integral over the positive integrand, $l_0 > 0$. If λ is larger than L/l_0 , there are no classical BPS configurations coming from the finite interval. If we do the same on the semi-infinite interval, then we will still get the contribution from the region around infinity, where $l(s)$ tends to zero, and this contribution is always 1.

2.4 Multidimensional Generalization

The consideration above can be generalized to the multidimensional case as follows. If the manifold M is compact, then the space \mathcal{S} of the periodic trajectories of Eq. (6) is bound by trajectories that pass through the critical points of dW . Thus, the function $l(s)$, being positive inside \mathcal{S} and tending to $+\infty$ at the boundaries, has a nonzero minimum l_0 , and, as in the previous subsection, we can get rid of the classical BPS configurations by rescaling the Kähler metric. Therefore, due to the CFIV index argument, there will be no quantum BPS states.

Now, suppose that M is noncompact, and it is obtained from the compact manifold \bar{M} by cutting out several submanifolds N_α of complex codimension 1. For the sake of simplicity, in this paper we will restrict ourselves to the case where submanifolds N_α are smooth and do not intersect with each other,

$$M = \bar{M} \setminus \cup N_\alpha. \quad (18)$$

Suppose that the 1-differential dW has simple poles on these submanifolds,

$$\int_{\Gamma_\alpha} dW = \Delta_\alpha W, \quad (19)$$

where Γ_α is a cycle in the vicinity of N_α that cannot be contracted to a point without crossing N_α , i.e. it has a nontrivial linking with N_α .

We claim that the number of the BPS states coming from the quantization of the space C_α of the parametrized curves $\Phi^i(x)$ that could be deformed to Γ_α is bound from below by the Euler number of N_α .

Let us study the space S_α of periodic trajectories in the space C_α . Rescaling the Kähler metric

$$G_{i\bar{j}} \rightarrow \lambda G_{i\bar{j}}$$

and taking λ to $+\infty$ forces $l(s)$ to go to $+\infty$ everywhere except for the curves in the vicinity of N_α . Thus, the problem is reduced to the vicinity of N_α that looks as $C^* \times N_\alpha$, and the differential is $cd\Phi/\Phi$, where Φ is the coordinate on $C^* = C \setminus 0$. Let us take the metric G in the vicinity of N_α in the following form:

$$G^{i\bar{j}} = G_{\text{pr}}^{i\bar{j}} + tg^{i\bar{j}} + O(t^2), \quad (20)$$

where $G_{\text{pr}}^{i\bar{j}}$ is the inverse product metric, $g^{i\bar{j}} = 0$ if $i > 1, \bar{j} > 1$ or $i = \bar{j} = 1$; $g^{i\bar{1}} = v^i$ and v^i is a vector field on N_α . For small t the trajectory starting at the point (Φ, P) , whose projection on C^* is periodic, will be nonperiodic in the N_α direction. The shift in this direction is proportional to $v^i(P)$, i.e. to the value of the vector field v^i at the point P on N_α . Thus, the number of the periodic trajectories is given by the number of the zeroes of the vector field v^i , i.e. it is bound from below by the Euler number of N_α .

3 A Typical Example

In this section we will consider a typical example that contains most of various cases considered in Sec. 2. Consider

$$K(\Phi, \bar{\Phi}) = \Phi \bar{\Phi}, \quad dW = \frac{4\pi}{2 - \cos \Phi} d\Phi, \quad (21)$$

The target space has the topology of a cylinder with two points deleted (Fig. 1),

$$-\infty < \text{Im}\Phi < \infty, \quad -\pi \leq \text{Re}\Phi \leq \pi, \quad (22)$$

and

$$(\Phi_*)_{1,2} = \pm i \ln(2 + \sqrt{3}). \quad (23)$$

Correspondingly, there are three noncontractible cycles, Γ_1, Γ_2 and Γ_3 in Fig. 1. The scalar potential is depicted in Fig. 2. Later on we will deform the Kähler potential in Eq. (21) by adding a small perturbation. Since $dW/d\Phi$ vanishes only at $|\text{Im}\Phi| \rightarrow \infty$, the model has the run-away vacua. The soliton solutions stabilize the theory, as in Ref. [13]. The periods corresponding to the cycles Γ_1, Γ_2 and Γ_3 are all equal to

$$\Delta W = \frac{8\pi^2}{\sqrt{3}}. \quad (24)$$

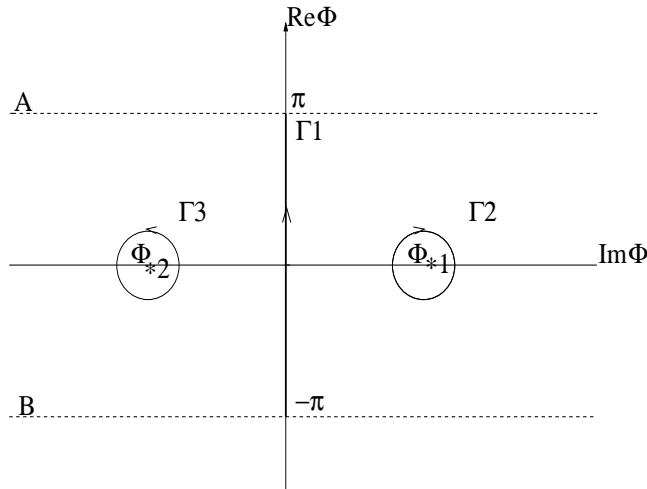


Figure 1: The target space in the problem (21). The dashed lines are glued, so that the points A and B are actually one and the same point.

In the problems with one variable Φ , the fact of existence of the integral of motion (13) is extremely helpful. By inspecting Fig. 1, one immediately infers that one can expect three solutions of Eq. (6) – one connecting the points $\Phi = -\pi$ and $\Phi = \pi$ along the real axis of Φ , and two other solutions winding around $(\Phi_*)_{1,2}$. Let us discuss them in turn.

For real Φ both equations in (6) coincide. The solution $\Phi(x, s)$ is readily obtained in an implicit form. It is given by inversion of the formula

$$x - \frac{1}{2} = \frac{\Phi}{2\pi} - \frac{1}{4\pi} \sin \Phi. \quad (25)$$

Here, $\delta = 0$, which follows from Eq. (24). And we omit the index s since in this case $s = I = 0$, just a fixed number. In the problem at hand each trajectory is in one-to-one correspondence with the value of I . So, we will label the trajectories by the corresponding value of I instead of s in this section. We obviously have $l(I = 0) = 1$, (as noted before, we set $L = 1$ here for convenience). The function $\Phi(x)$ is depicted in Fig. 3, while the energy density corresponding to this solution is plotted in Fig. 4. Of course, the center of the soliton ($1/2$ in Eq. (25)) can be chosen arbitrarily. As we will see shortly, this trajectory is exceptional.

The solitons corresponding to $\Gamma_{2,3}$ can be established as follows. We will focus on Γ_2 since the solution for Γ_3 is similar. The superpotential is obtained as

$$W = \frac{8\pi}{\sqrt{3}} \arctan \left(\sqrt{3} \tan \frac{\Phi}{2} \right). \quad (26)$$

Here we have chosen a specific branch of the multivalued superpotential W . Correspondingly, for the trajectories winding around $(\Phi_*)_1$ the value of $I = \text{Im } W$ spans

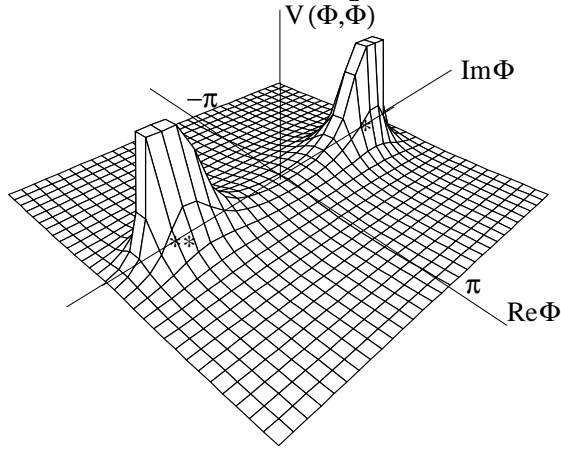


Figure 2: The scalar potential $V(\Phi, \bar{\Phi})$ in the problem (21) near $(\Phi_*)_{1,2}$. Here * denotes Φ_{*1} , ** denotes Φ_{*2} .

the interval $I = (0, \infty)$. Following (14), we can get $\Phi(x, I)$ implicitly from

$$\text{Im}W(\Phi_I) = I.$$

Then, we can parametrize the corresponding closed curve Γ_I as $\Phi_I(y)$, $0 < y < 1$ as in Sec. 2. Then, from Eq. (8), we can get $l(I)$. Let us write

$$\Phi = \phi + i\chi$$

where ϕ and χ are real functions of x . The condition

$$I(\phi, \chi) = \text{a positive constant}$$

defines a family of trajectories (see Sec. 2). Practically, for each given trajectory one can find analytically $\phi_I(x)$ and $\chi_I(x)$ using the BPS saturation equation

$$\dot{\Phi} = \frac{4\pi}{2 - \cos \Phi}, \quad (27)$$

and then obtain $l(I)$. The period function $l(I)$ *versus* I is shown in Fig. 6. The solution corresponding to the BPS soliton is obtained from the condition $l(I_0) = 1$. The energy density corresponding to the Γ_2 soliton solution is shown in Fig. 5. Note that we denote the class of the trajectories homotopical to the Γ_1 cycle as the class T_1 , such as the trajectories Γ', Γ'' in Fig. 7 below, and the classes of the trajectories homotopical to the Γ_2, Γ_3 cycles as the class T_2, T_3 . And we know that the class T_3 could be treated absolutely in the same way as the class T_2 .

Thus, classically we have three BPS soliton solutions preserving one half of supersymmetry. Let us see what happens in the weak coupling regime. To this end we introduce a small coupling constant g^2 in the Kähler metric,

$$K(\Phi, \bar{\Phi}) = \frac{1}{g^2} \Phi \bar{\Phi}. \quad (28)$$

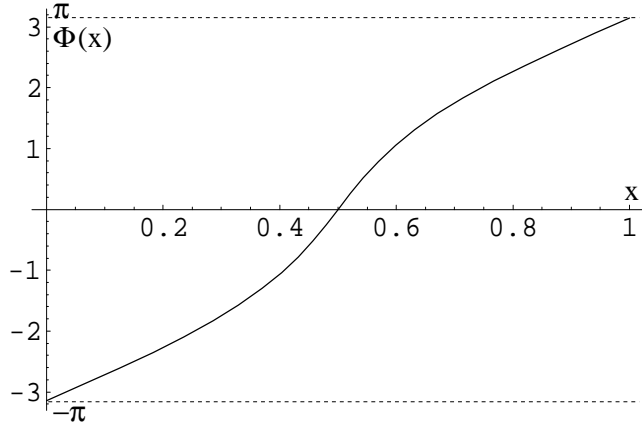


Figure 3: Φ versus x for the trajectory running along Γ_1 .

One can view the factor $1/g^2$ as a deformed metric. The equation of the BPS saturation remains the same as in Eq. (27), if, instead of the original variable x , one introduces $\tilde{x} = g^2 x$. The dot will now denote differentiation over \tilde{x} . The solution is of the same type as discussed above with the constraint that the period in \tilde{x} is $g^2 \ll 1$ rather than unity. The BPS saturation equations have no solution with the small period for the trajectory along Γ_1 . (We hasten to note, however, that a nonsaturated solution of the classical equations of motion, winding around Γ_1 with the arbitrary period, always exists, as it is perfectly clear from the examination of Fig. 1.)

The BPS saturation equations do have small period solutions for the $\Gamma_{2,3}$ cycles. The requirement $l(I) = g^2 \rightarrow 0$ implies that the trajectories wind around $(\Phi_*)_{1,2}$ very close to $(\Phi_*)_{1,2}$ (see Fig. 6 and 7). Then, say for the Γ_2 cycle, $dW/d\Phi$ can be replaced by

$$\frac{dW}{d\Phi} = \frac{4\pi}{\sqrt{3}i} \frac{1}{\Phi - (\Phi_*)_1}, \quad (29)$$

and the solution with the given l takes the form

$$\Phi = (\Phi_*)_1 + \left(\frac{2l}{\sqrt{3}} \right)^{1/2} \exp \left(\frac{2\pi i \tilde{x}}{l} \right), \quad l = g^2. \quad (30)$$

This is precisely the solution which was first obtained in Ref. [13].

As was mentioned, the existence of the solution preserving one half of SUSY at small coupling immediately translates at the quantum level into the presence in the spectrum of the BPS multiplet of particles. That such particles neither appear nor disappear in the process of evolution from small g^2 to $g^2 = 1$ is guaranteed by the CFIV index. At this point, one may think that there are three BPS solutions at the classical level and two BPS solutions at the quantum level in this example. But it was said in Sec. 2.3 that the number of doublets in the quantum theory equals the

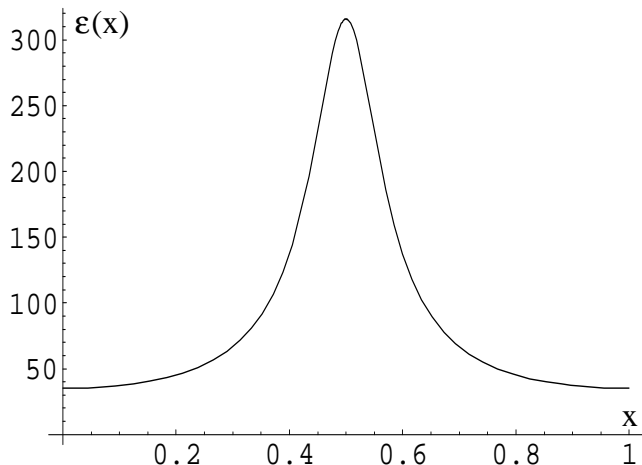


Figure 4: The energy density $\epsilon(x) = \dot{\Phi}(x)^2 + V[\Phi(x)]$ versus x for the Γ_1 soliton solution.

number of quasiclassical doublets only mod 2, which, at first sight, contradicts this example. However, one should notice that the classical BPS solution corresponding to the cycle Γ_1 presents actually a degeneration of two solutions that are “glued” together, which can be seen clearly from Fig. 6 that the two solutions Γ', Γ'' are “glued” together at the point corresponding to the Γ_1 solution, so that the relative fermion charges of these two solutions are arranged in such a way that their contributions to the CFIV index cancel each other. Therefore, this example is indeed in agreement with the analysis of Sec. 2.3.

Summarizing, in the given sample problem, with three noncontractible cycles in the target space, we found three distinct soliton supermultiplets. Two of them, corresponding to the trajectories winding around the poles of $dW/d\Phi$, are BPS saturated (short multiplets, one half of supersymmetry is preserved). The soliton corresponding to the Γ_1 cycle is not saturated, in spite of the fact that classically one can find a solution of the BPS saturation equations in strong coupling at certain (isolated) values of g^2 . The classical solution is not elevated to the quantum level.

4 The Quantum Mechanical Explanation of Instantonic Corrections to the Quasiclassical BPS States

Consideration of Sec. 2 may leave a wrong impression that the phenomenon of localization to the poles of dW in the computation of the number of the BPS doublets is quite specific to $d = 2$ theories. In this section we will show that (as it always happens with phenomena in the computation of various index-like quantities) this

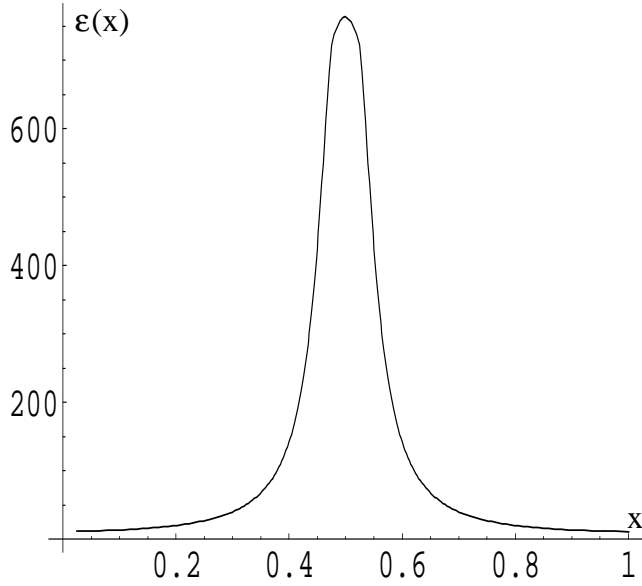


Figure 5: The energy density $\epsilon(x)$ versus x for the Γ_2 soliton solution.

phenomenon is a special case of a more general phenomenon that occurs in $\mathcal{N} = 2$ quantum mechanics with the target space with isometry (the Killing vector). Moreover, we will show that the mysterious nonperturbative corrections mentioned in Sec. 2 could be easily understood as instantonic corrections in Witten's $\mathcal{N} = 1$ quantum mechanics associated with the $\mathcal{N} = 2$ quantum mechanics on the target space with isometry.

4.1 Doublets of $\mathcal{N} = 2$ ($d = 2$) Superalgebra and Ground States of Associated Quantum Mechanics

The algebra of supersymmetries of the $\mathcal{N} = 2$ two-dimensional field theory has the following form:

$$\{Q_{\pm}, \bar{Q}_{\pm}\} = H \pm P, \quad \{Q_+, Q_-\} = Z, \quad \{\bar{Q}_+, \bar{Q}_-\} = \bar{Z}. \quad (31)$$

Here P is the momentum operator in the x direction, H is the Hamiltonian, the operators Z and \bar{Z} are complex conjugate to each other and are called the central charges. The subscripts $+$ and $-$ of the supercharges denote that they have the charges $+1/2$ and $-1/2$ under the $SO(1,1)$ Lorentz group. Operators without the bar are the chiral supercharges, while those with the bar are antichiral ones.

To simplify the study of the representations of the algebra (31), we will make the following redefinition:

$$q_1 = Q_+, \quad \bar{q}_1 = \bar{Q}_+, \quad q_2 = \bar{Q}_-, \quad \bar{q}_2 = Q_-. \quad (32)$$

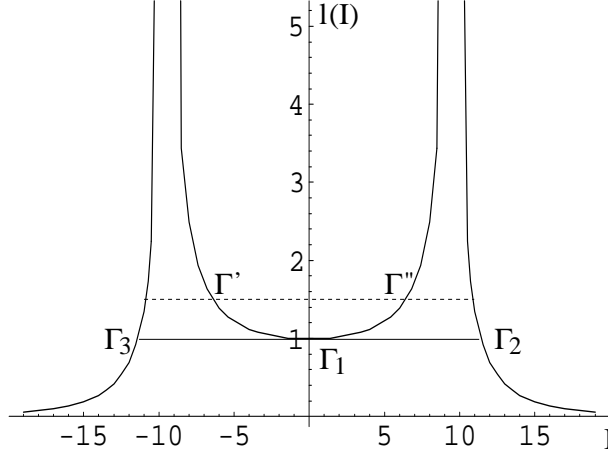


Figure 6: The period $l(I)$ versus I .

Then the commutation relations (31) take a well-known form of the Clifford algebra

$$\{q_\alpha, \bar{q}_{\bar{\beta}}\} = N_{\alpha\bar{\beta}}, \quad (33)$$

where $\alpha, \bar{\beta} = 1, 2$ and the 2×2 matrix N is

$$N = \begin{bmatrix} H + P & Z \\ \bar{Z} & H - P \end{bmatrix}. \quad (34)$$

We are interested in the states that represent particles at rest, so we will restrict ourselves to the representations with $P = 0$.

In the irreducible representations of the algebra (33), H and Z are represented by numbers E and Z . We are interested in such representations that $Z \neq 0$. Then the irreducible representation is either 4-dimensional, if the matrix N is nondegenerate, or two dimensional, if the matrix N has a zero eigenvalue. The latter happens if and only if

$$Z = e^{i\theta} E, \quad (35)$$

where θ is a real constant. The shortening of the irreducible representation could also be interpreted as follows. The $\mathcal{N} = 2$ algebra contains, as a subalgebra, the following algebra of an associated $\mathcal{N} = 1$ quantum mechanics:

$$\{q_\theta, \bar{q}_\theta\} = H_\theta, \quad (36)$$

where

$$q_\theta = q_1 + e^{i\theta} q_2, \quad \bar{q}_\theta = \bar{q}_1 + e^{-i\theta} \bar{q}_2, \quad H_\theta = 2H + e^{-i\theta} Z + e^{i\theta} \bar{Z}. \quad (37)$$

Then, the doublet representations of the algebra (33) are in one-to-one correspondence with the ground states of the associated $\mathcal{N} = 1$ quantum mechanics (36).

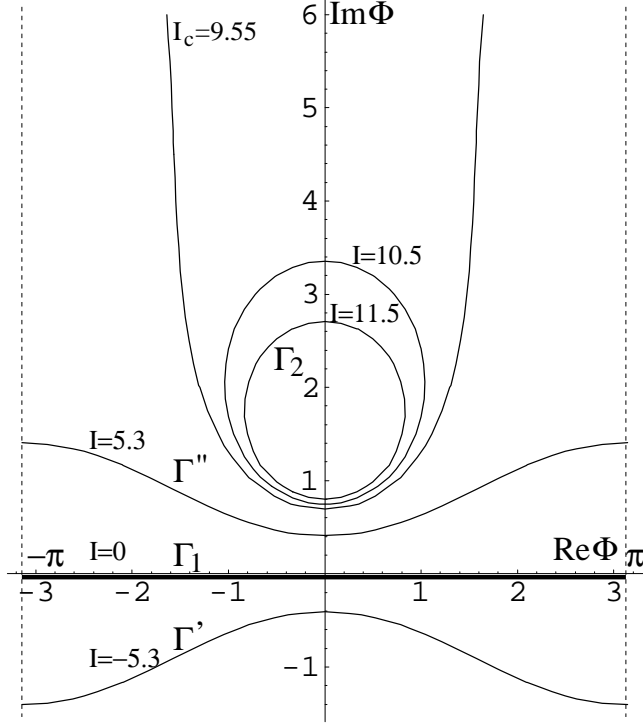


Figure 7: The typical trajectories for the homotopy classes T_1 and T_2 . Here, I_c is the corresponding value of I for which the period $l(I)$ goes to infinity. And the dashed lines are glued.

4.2 The Representation of the $\mathcal{N} = 2$ ($d = 2$) Algebra from the Representation of the $\mathcal{N} = 2$ Quantum Mechanics on the Target Space with an Isometry

Let us start from the conventional superalgebra of $\mathcal{N} = 2$ quantum mechanics

$$\{q_{1,0}, \bar{q}_{1,0}\} = \{q_{2,0}, \bar{q}_{2,0}\} = H. \quad (38)$$

The $\mathcal{N} = 2$ quantum mechanics, obtained by dimensional reduction of the generalized Landau-Ginzburg theory, provides the following representation of the algebra (38).

Consider the Clifford algebra

$$\{\psi_+^i, \bar{\psi}_+^{\bar{j}}\} = G^{i\bar{j}}; \quad \{\psi_-^i, \bar{\psi}_-^{\bar{j}}\} = G^{i\bar{j}}. \quad (39)$$

Then, the algebra (38) has the following representation:

$$Q_{+,0} = q_{1,0} = \bar{\psi}_-^{\bar{j}} \frac{\partial}{\partial \bar{\Phi}^{\bar{j}}} + \psi_+^j \Omega_j - \psi_+^l \bar{\psi}_-^{\bar{i}} \bar{\psi}_+^{\bar{j}} \frac{\partial G_{l\bar{j}}}{\partial \bar{\Phi}^{\bar{i}}},$$

$$\begin{aligned}
Q_{-,0} &= \bar{q}_{2,0} = \bar{\psi}_+^{\bar{j}} \frac{\partial}{\partial \bar{\Phi}^{\bar{j}}} + \psi_-^j \Omega_j - \psi_-^l \bar{\psi}_-^{\bar{i}} \bar{\psi}_+^{\bar{j}} \frac{\partial G_{l\bar{j}}}{\partial \bar{\Phi}^{\bar{i}}} , \\
\bar{Q}_{+,0} &= \bar{q}_{1,0} = \psi_-^j \frac{\partial}{\partial \Phi^j} + \bar{\psi}_+^{\bar{j}} \bar{\Omega}_{\bar{j}} - \bar{\psi}_+^{\bar{l}} \psi_-^i \psi_+^j \frac{\partial G_{j\bar{l}}}{\partial \Phi^i} , \\
\bar{Q}_{-,0} &= q_{2,0} = \psi_+^j \frac{\partial}{\partial \Phi^j} + \bar{\psi}_-^{\bar{j}} \bar{\Omega}_{\bar{j}} - \bar{\psi}_-^{\bar{l}} \psi_-^i \psi_+^j \frac{\partial G_{j\bar{l}}}{\partial \Phi^i} .
\end{aligned} \tag{40}$$

Suppose that the target space has an isometry that preserves both the metric and the one-differential. It means that there is a real Killing vector field ³

$$v^m = (v^i, \bar{v}^{\bar{i}}) ,$$

and the Lie derivative along this real vector field $L_v + \bar{L}_{\bar{v}}$ leaves both the metric G and the differential Ω invariant. Here L_v acts on the (k, l) tensor as follows:

$$L_v T_{i_1 \dots i_k, \bar{j}_1, \dots, \bar{j}_l} = v^j \frac{\partial}{\partial \Phi^j} T_{i_1 \dots i_k, \bar{j}_1, \dots, \bar{j}_l} + \sum_{a=1}^k T_{i_1 \dots i_{a-1} i \dots i_k, \bar{j}_1, \dots, \bar{j}_l} \frac{\partial v^i}{\partial \Phi^{i_a}} , \tag{41}$$

and $\bar{L}_{\bar{v}}$ is complex conjugated to L_v .

This invariance means that

$$\left(v^j \frac{\partial}{\partial \Phi^j} + \bar{v}^{\bar{j}} \frac{\partial}{\partial \bar{\Phi}^{\bar{j}}} \right) G_{i\bar{k}} + G_{j\bar{k}} \frac{\partial v^j}{\partial \Phi^i} - G_{i\bar{j}} \frac{\partial \bar{v}^{\bar{k}}}{\partial \bar{\Phi}^{\bar{j}}} = 0 , \tag{42}$$

$$v^j \frac{\partial}{\partial \Phi^j} \Omega_i + \Omega_j \frac{\partial v^j}{\partial \Phi^i} = 0 . \tag{43}$$

Note, that the antiholomorphic derivatives are absent in (43) due to holomorphy of Ω .

Using this vector field, we can modify the representation of $\mathcal{N} = 2$ supersymmetric quantum mechanics in such a way that it represents the algebra of $\mathcal{N} = 2$ ($d = 2$) supersymmetry in the following way:

$$\begin{aligned}
q_1 = Q_+ &= Q_{+,0} + \frac{1}{2} G_{i\bar{j}} v^i \bar{\psi}_-^{\bar{j}} , \\
\bar{q}_2 = Q_- &= Q_{-,0} + \frac{1}{2} G_{i\bar{j}} v^i \bar{\psi}_+^{\bar{j}} , \\
\bar{q}_1 = \bar{Q}_+ &= \bar{Q}_{+,0} + \frac{1}{2} G_{i\bar{j}} \bar{v}^{\bar{j}} \psi_-^i , \\
q_2 = \bar{Q}_- &= \bar{Q}_{-,0} + \frac{1}{2} G_{i\bar{j}} \bar{v}^{\bar{j}} \psi_+^i .
\end{aligned} \tag{44}$$

Here P is represented as

$$P \rightarrow (L_v + \bar{L}_{\bar{v}}) , \tag{45}$$

³Here and below we will assume that m is a real index, i.e. it can take both holomorphic (i), and antiholomorphic (\bar{i}) values.

while the central term

$$Z = v^i \Omega_i. \quad (46)$$

Note that Eq. (43) implies that Z is a constant.

Now, let us have a closer look at the associated $\mathcal{N} = 1$ quantum mechanics. In particular, if we introduce

$$\begin{aligned} \chi_1^i &= e^{i\theta} \psi_+^i, & \bar{\chi}_1^{\bar{i}} &= \bar{\psi}_-^{\bar{i}} \\ \chi_2^i &= \psi_-^i, & \bar{\chi}_2^{\bar{i}} &= e^{-i\theta} \bar{\psi}_+^{\bar{i}}, \end{aligned}$$

then the supercharge q_θ takes the form:

$$q_\theta = \chi_1^m \left(\frac{\partial}{\partial \Phi^m} + \omega_m(\Phi, \bar{\Phi}) \right), \quad (47)$$

where $\omega_m = (\omega_i, \bar{\omega}_{\bar{i}})$, and

$$\begin{aligned} \omega_i &= e^{-i\theta} \Omega_i + \frac{1}{2} G_{i\bar{j}} \bar{v}^{\bar{j}} - \bar{\psi}_-^{\bar{l}} \psi_-^j \frac{\partial G_{i\bar{l}}}{\partial \Phi^j}, \\ \bar{\omega}_{\bar{i}} &= e^{i\theta} \bar{\Omega}_{\bar{i}} + \frac{1}{2} G_{i\bar{i}} v^i - \bar{\psi}_+^{\bar{j}} \psi_+^l \frac{\partial G_{l\bar{j}}}{\partial \bar{\Phi}^{\bar{i}}}. \end{aligned} \quad (48)$$

One can check that ω is closed, i.e.

$$\frac{\partial \omega_m}{\partial \Phi^n} = \frac{\partial \omega_n}{\partial \Phi^m}.$$

(Here, as above, $\Phi^m = (\Phi^i, \bar{\Phi}^{\bar{i}})$.) In the computation of Witten's index one can continuously change the superpotential (in a way preserving discretization of the spectrum). Therefore, we can study a family of the $\omega_i(\lambda)$ as the following

$$\omega_i(\lambda) = e^{-i\theta} \Omega_i + \frac{\lambda}{2} \bar{v}^{\bar{j}} G_{i\bar{j}} - \bar{\psi}_-^{\bar{l}} \psi_-^j \frac{\partial G_{i\bar{l}}}{\partial \Phi^j}. \quad (49)$$

As it is well known, the classical ground states correspond to zeroes of $\omega(\lambda)$. Let us tend λ to $+\infty$. At first, suppose that target space is compact. Then, the zeroes of $\omega(\lambda)$ tend to the zeroes of v , and if v has no zeroes, Witten's index is equal to zero. Now, suppose that the target space is not compact, and Ω has poles on its compactification. Then, as λ tends to $+\infty$, the position of the zeroes of $\omega(\lambda)$ tends to the position of the poles, and the computation of Witten's index is reduced to the computation in the vicinity of the poles.

4.3 $\mathcal{N} = 2$ ($d = 2$) Field Theory as $\mathcal{N} = 2$ Quantum Mechanics on the Loop Space

The two-dimensional generalized Landau-Ginzburg model, as the quantum theory in the given winding sector, is an $\mathcal{N} = 2$ quantum mechanics on the corresponding

loop space modified by the vector field that rotates the loop. Various aspects of this phenomena were studied previously in [14, 15, 16, 17].

The coordinates on the loop space are

$$\Phi^{i,x} = \Phi^i(x).$$

The vector field $v^{i,x}$ generated by $x \rightarrow x + \epsilon$ is

$$v^{i,x} = \frac{\partial \Phi^i(x)}{\partial x}. \quad (50)$$

Let us see how this modifies the supercharges. For example, Q_+ takes the following form (the sum over the continuous index x is replaced by the integral)

$$Q_+ = \int dx \left\{ \bar{\psi}^{\bar{j}}_-(x) \left(\frac{\partial}{\partial \bar{\Phi}^{\bar{j}}}(x) + \frac{1}{2} G_{i\bar{j}} \frac{\partial \Phi^i(x)}{\partial x} \right) + \psi^j_+(x) \Omega_j(\Phi(x)) - \psi^l_+ \bar{\psi}^{\bar{i}}_- \bar{\psi}^{\bar{j}}_+ \frac{\partial G_{l\bar{j}}}{\partial \bar{\Phi}^{\bar{i}}} \right\}. \quad (51)$$

However, this is exactly the supercharge of the $d = 2$ generalized Landau–Ginzburg field theory.

Now, we check how the formula for the central charge works in this formalism,

$$Z = \sum_{i,x} v^{i,x} \Omega_{i,x} = \int \frac{dW}{d\Phi} \frac{d\Phi}{dx} dx, \quad (52)$$

in full agreement with the computation performed in Sec. 2. One can check that the zeros of the derivative of the superpotential of the associated quantum mechanics are nothing but the closed BPS trajectories!

So, we conclude that the nonperturbative phenomena that were anticipated in Sec. 2 actually exist in the form of the Witten instantonic transition in the associated $\mathcal{N} = 1$ quantum mechanics on the loop space.

5 Conclusions

In the theories with the large extra dimensions [4] – a popular subject of theoretical studies at present – one has to deal with the domain walls on the manifolds of the cylinder type. The issue of the BPS saturation *versus* nonsaturation of these domain walls is of the paramount importance. The dynamical part of this problem obviously reduces to the analysis of two-dimensional field theory with the extended supersymmetry on $R \times S$.

Here we addressed the dynamical question of the existence of the BPS saturated states within the framework of the generalized Wess–Zumino (or Landau–Ginzburg) models, describing the interaction (possibly, effective) of one or more chiral superfields. Since in such models the central charge $Z \propto \Delta W$, nonvanishing central charges are impossible, at first sight. We explained where this naive point of view is

wrong, and presented the theory of the BPS saturated states, both at the classical and quantum levels.

We revealed nonperturbative effects lifting the BPS saturation for the classical BPS solutions. It is shown that at the quantum (nonperturbative) level the BPS states can exist only if the target space of the Landau-Ginzburg model considered is noncompact. Using various index-related tools we found the number of the quantum BPS particles in the theories with one chiral superfield, and the lower bound on this number for more than one chiral superfield.

6 Acknowledgments

A part of this work was done while one of the authors (M.S.) was visiting the Aspen Center for Physics, within the framework of the program *Phenomenology of Superparticles and Superbranes*, in July 1999. Preliminary results were reported at this workshop. We would like to thank G. Dvali and G. Gabadadze for useful discussions.

The work was supported in part by DOE under the grant number DE-FG02-94ER40823. A.L. acknowledges financial support extended to him during his stay at Theoretical Physics Institute, University of Minnesota. The work of A.L. was also supported in part by the Grant for Support of Scientific Schools number 96-15-96455 and by the RFFI grant number 98-01-00328.

Appendix: The CFIV Index is Independent of the D Terms

The conventional Witten's index is known to be independent of any small smooth deformations of the supersymmetric theory. This is not true for the CFIV index. Still, this index is independent of the continuous variations of the D terms, i.e. terms in the Hamiltonian that are equal to

$$\{Q_+, [\bar{Q}_-, R]\} = \{q_1, [q_2, R]\}$$

or to

$$\{Q_-, [\bar{Q}_+, \bar{R}]\} = \{\bar{q}_2, [\bar{q}_1, \bar{R}]\}$$

with some operators R and \bar{R} . Say, a small variation of R , $\delta H = \{q_1, [q_2, \delta R]\}$, results in the following variation of the CFIV index:

$$\begin{aligned} \delta I_{\text{CFIV}} &= \text{tr} (-1)^F F \exp(-tH) \{Q_+, [\bar{Q}_-, \delta R]\} (-t) \\ &= \text{tr} (-1)^F F \exp(-tH) \left(Q_+ [\bar{Q}_-, \delta R] + [\bar{Q}_-, \delta R] Q_+ \right) (-t). \end{aligned} \quad (\text{A.1})$$

Now, we take Q_+ in the second term, put it to the leftmost position inside the trace, and then drag to the right, using the fact that

$$Q_+(-1)^F F = -(-1)^F Q_+ F = -(-1)^F Q_+ - (-1)^F F Q_+.$$

In this way we get

$$\delta I_{\text{CFIV}} = -\text{tr}(-1)^F \exp(-tH) Q_+ [\bar{Q}_-, \delta R](-t). \quad (\text{A.2})$$

Repeating the same operation with \bar{Q}_- yields $\delta I_{\text{CFIV}} = 0$ since \bar{Q}_+ anticommutes with Q_- .

References

- [1] V.A. Rubakov and M.E. Shaposhnikov, *Phys. Lett.* **125B**, 136 (1983).
- [2] G. Dvali and M. Shifman, *Nucl. Phys.* **B504**, 127 (1997) [hep-th/9611213].
- [3] G. Dvali and M. Shifman, *Phys. Lett.* **B396**, 64 (1997) [hep-th/9612128]; Erratum **B407**, 452 (1997).
- [4] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett.* **B429**, 263 (1998) [hep-ph/9803315]; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett.* **B436**, 257 (1998) [hep-ph/9804398].
- [5] Z. Kakushadze, *Nucl. Phys.* **B548**, 205 (1999) [hep-th/9811193]; *Nucl. Phys.* **B552**, 3 (1999) [hep-th/9812163]; *Nucl. Phys.* **B551**, 549 (1999) [hep-th/9902080]; N. Arkani-Hamed, S. Dimopoulos, G. Dvali and J. March-Russell, hep-ph/9811448; E. Faraggi and M. Pospelov, *Phys. Lett.* **B458**, 237 (1999) [hep-ph/9901299]; N. Arkani-Hamed and M. Schmaltz, hep-ph/9903417; G. Dvali and A. Smirnov, hep-ph/9904211; G. Dvali and G. Gabadadze, *Phys. Lett.* **B460**, 47 (1999) [hep-ph/9904221]; L. Hall and C. Kolda, *Phys. Lett.* **B459**, 213 (1999) [hep-ph/9904236].
- [6] B. Chibisov and M. Shifman, *Phys. Rev.* **D56**, 7990 (1997) [hep-th/9706141]; Erratum **D58**, 109901 (1998).
- [7] S. Cecotti and C. Vafa, *Commun. Math. Phys.* **158**, 569 (1993) [hep-th/9211097].
- [8] E. Witten and D. Olive, *Phys. Lett.* **B78**, 97 (1978).
- [9] E. Witten, *J. Diff. Geom.* **17**, 661 (1982).
- [10] S. Cecotti, P. Fendley, K. Intriligator and C. Vafa, *Nucl. Phys.* **B386**, 405 (1992) [hep-th/9204102].

- [11] P. Fendley and K. Intriligator, *Nucl. Phys.* **B372**, 533 (1992).
- [12] E. Witten, *Nucl. Phys.* **B202**, 253 (1982).
- [13] G. Dvali and M. Shifman, *Phys. Lett.* **B454**, 277 (1999) [hep-th/9901111].
- [14] S. Cecotti, L. Girardello, A. Pasquinucci, *Int. J. Mod. Phys.* **A6**, 2427 (1991).
- [15] N. Nekrasov, *Four-dimensional holomorphic theories*, PhD Thesis, Princeton University, UMI-9701221.
- [16] N. Nekrasov, *Nucl. Phys.* **B531**, 323 (1998) [hep-th/9609219].
- [17] L. Baulieu, A. Losev, and N. Nekrasov, *Nucl. Phys.* **B522**, 82 (1998) [hep-th/9707174].